MRC Performance for M-ary Modulation in Arbitrarily Correlated Nakagami Fading Channels

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***Abstract*—We derive the symbol error probability for coherent detection of several types of M-ary modulation schemes using maximal ratio combining. We consider Nakagami fading channels, where the instantaneous signal-to-noise ratios of the diversity branches are not necessarily independent or identically distributed. The proposed problem is made analytically tractable by transforming the correlated physical diversity branches into independent *virtual branches*.**

***Index Terms*—Correlated Nakagami fading, diversity combining, error probability, maximal ratio combining.**

I. Introduction

NAKAGAMI fading channels have received considerable attention in the study of wireless systems [1]–[4]. Results have shown that the Nakagami distribution (also known as the "m-distribution") provides greater flexibility in matching experimental data collected in a variety of fading environments [5].

Analysis of maximal ratio combining (MRC) in Nakagami fading has typically assumed that the diversity branches are independent [1], [3]. Consideration of correlated fading has been limited only to dual-branch diversity (see, for example, [1]), with the exception of [4], [6]. The studies in [4], [6], though, considered only binary modulation with the assumption of equal m as well as equal average signal-to-noise ratios (SNR's) among all diversity branches.

However, in some cases the average SNR is not necessarily equal for all branches and the fading statistics can also be different for each diversity branch. Examples of unequal average SNR and/or fading statistics include: 1) angle diversity using multiple beams where the average signal strength and fading statistics can be different in each beam; 2) polarization diversity with high base station antennas where for a vertically-polarized transmitter, the average received signal strength at the horizontally-polarized antenna is typically 6–10 dB lower than at the vertically-polarized antenna; 3) macrodiversity, where the shadow fading is different at each antenna and different local scattering conditions can lead to different fading statistics; and 4) Rake receivers, where the distribution of signal power at different delay is not uniform and the first arriving multipath component is more likely to be specular than the later components. In these cases, closed-form expressions for the performance of MRC are not previously available in the literature.

In this paper, we derive exact symbol error probability (SEP) expressions for coherent detection of several types of M-ary modulation schemes using MRC with an arbitrary number of diversity branches. We consider correlated Nakagami fading channels with integer-order fading parameters, where: 1) the instantaneous SNR's can be arbitrarily correlated; 2) the SNR distributions can be from different Nakagami families, i.e., the fading parameters m are not necessarily equal; and 3) the average SNR's (averaged over the fading) of the branches are not necessarily equal.

II. Diversity Combining Analysis

A. Preliminaries

With MRC, the received signals from multiple diversity branches are cophased, weighted, and combined to maximize the output SNR. The instantaneous output SNR is given as $\gamma_{\mathrm{MRC}} = \sum_{i=1}^{N} \gamma_i$, where N is the number of available diversity branches and γ_i denotes the instantaneous SNR of the ith diversity branch. The instantaneous SNR can be expressed as $\gamma_i \triangleq \alpha_i^2 E_{\mathrm{s}}/N_{0i}$, where E_{s} is the average symbol energy, and α_i is the instantaneous fading amplitude and N_{0i} is the noise power spectral density of the ith diversity branch.

For a correlated Nakagami fading channel, the marginal probability density function (p.d.f.) of α_i is Nakagami distributed and thus the marginal p.d.f. of γ_i is given by[1]

$$g_{\gamma_i}(x) = \frac{1}{\Gamma(m_i)} \left(\frac{m_i}{\Gamma_i}\right)^{m_i} x^{m_i - 1} e^{-m_i x/\Gamma_i} \tag{1}$$

where m_i denotes the Nakagami family, and the average SNR is $\Gamma_i = \mathbb{E}\{\gamma_i\}$. Note that the α_i's can be from different Nakagami families, where m_i and Γ_i are not necessarily equal among the branches.

B. The Virtual Branch Technique

Conventional analysis of MRC in correlated Nakagami fading is, in general, cumbersome and complicated. This is alleviated in the following by transforming the dependent physical branch variables into a new set of independent *virtual branches* and expressing the combiner output SNR as a linear function of the independent virtual branch SNR's.

Without loss of generality, one can assume that the γ_i's are indexed in increasing order of their Nakagami parameters, i.e.,

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[1]The notation $\Gamma(\cdot)$ is used to denote the gamma function, and Γ_i denotes the average SNR of the ith branch. This notation is chosen to be consistent with the literature, and the distinction should be clear from the context.

$m_1 \leq m_2 \leq \cdots \leq m_N$. Let X_i be $2m_i \times 1$ vectors defined by $X_i \triangleq [X_{i,1} X_{i,2} \cdots X_{i,2m_i}]^t$, for $i = 1, 2, \cdots, N$, where $(\cdot)^t$ denotes transpose and the elements $X_{i,k}$ are independent and identically distributed (i.i.d.) Gaussian random variables with zero mean and variance $\mathbb{E}\{X_{i,k}^2\} = (\Gamma_i / 2m_i)$. We introduce the $D_T \times 1$ vector X defined by $X \triangleq [X_1^t \quad X_2^t \cdots X_N^t]^t$, where $D_T = \sum_{i=1}^N 2m_i$, with covariance matrix given by $K_X = \mathbb{E}\{XX^t\}$. The correlation among the elements of X is constructed such that

$$\mathbb{E}\{X_{i,k} X_{j,l}\} = \begin{cases} \frac{\Gamma_i}{2m_i} & \text{if } i = j \text{ and } k = l \\ \rho_{i,j} \sqrt{\frac{\Gamma_i}{2m_i} \frac{\Gamma_j}{2m_j}}, & \text{if } i \neq j \text{ but} \\ & k = l = 1, 2, \cdots \\ & \quad \times 2 \min\{m_i, m_j\} \\ 0, & \text{otherwise.} \end{cases}$$

It can be shown that the relationship between the covariance of γ_i and γ_j and the correlation of the elements of X is given by

$$\rho_{\gamma_i \gamma_j} \triangleq \frac{\mathbb{E}\{(\gamma_i - \Gamma_i)(\gamma_j - \Gamma_j)\}}{\sqrt{\text{var}\{\gamma_i\} \text{var}\{\gamma_j\}}} = \sqrt{\frac{\min\{m_i, m_j\}}{\max\{m_i, m_j\}}} \rho_{i,j}^2. \tag{2}$$

Let $\{\lambda_l\}$ be the set of L *distinct* eigenvalues of K_X where each λ_l has algebraic multiplicity μ_l such that $\sum_{l=1}^L \mu_l = D_T$. By using the Karhunen–Loève (KL) expansion of the vector X, the combiner output SNR can be written as[2] $\gamma_{\text{MRC}} \overset{L}{=} \sum_{l=1}^L \lambda_l V_l$, where the notation "$\overset{L}{=}$" denotes "equal in their respective distributions" (or "Laws") [9], and the virtual branch variables V_l are *independent* chi-squared random variables with μ_l degrees of freedom. Therefore the characteristic function (c.f.) of V_l is given by $\psi_{V_l}(j\nu) \triangleq \mathbb{E}\{e^{+j\nu V_l}\} = 1/(1 - 2j\nu)^{\mu_l/2}$.

III. SEP Over the Channel Ensemble

The SEP for MRC in correlated Nakagami fading is obtained as $P_e = \mathbb{E}_{\gamma_{\text{MRC}}}\{\Pr\{e|\gamma_{\text{MRC}}\}\}$, where for a general class of modulation schemes

$$\Pr\{e|\gamma_{\text{MRC}}\} = \sum_{k=1}^K \int_0^{\Theta_k} a_k(\theta) e^{-\phi_k(\theta)\gamma_{\text{MRC}}} \, d\theta \tag{3}$$

where $a_k(\theta)$, $\phi_k(\theta)$ and Θ_k are parameters particular to the specific modulation format and are independent of γ_{MRC}. Table I lists these parameters for some common coherent modulations:[3] M-ary phase shift keying (MPSK), M-ary square quadrature amplitude modulation (MQAM) with $M = 2^l$ and l even, M-ary pulse amplitude modulation (MPAM), binary frequency shift keying (BFSK), BFSK with minimum correlation (BFSK$_{\text{min}}$), coherent detection of differentially encoded BPSK (DE-BPSK), and precoded minimum shift keying (MSK) [10], [11].

[2]A similar technique employing a frequency-domain KL expansion was used in [7] to study diversity combining in a frequency-selective Rayleigh-fading channel. Another technique similar to the KL expansion was also used in [8] to study the reception of noncoherent orthogonal signals in Rician and Rayleigh fading channels.

[3]Since MRC requires channel phase estimates, it is generally used in conjunction with coherent modulation schemes. If channel phase estimates are not available, then one may resort to diversity combining techniques such as postdetection equal gain combining with noncoherent or differentially coherent modulation.

TABLE I
Parameters for Specific Modulation Schemes

	K	$a_k(\theta)\cdot\pi$	$\phi_k(\theta)\cdot\sin^2(\theta)$	Θ_k/π
MPSK	1	1	$\sin^2\left(\frac{\pi}{M}\right)$	$\left(1-\frac{1}{M}\right)$
MQAM	2	$4\left(1-\frac{1}{\sqrt{M}}\right)$	$\frac{3}{2(M-1)}$	1/2
		$-4\left(1-\frac{1}{\sqrt{M}}\right)^2$	$\frac{3}{2(M-1)}$	1/4
MPAM	1	$2\left(1-\frac{1}{M}\right)$	$\frac{3}{M^2-1}$	1/2
BFSK	1	1	$\frac{1}{2}$	1/2
BFSK$_{\text{min}}$	1	1	$\left(\frac{1}{2}+\frac{1}{3\pi}\right)$	1/2
DE-BPSK	2	2	1	1/2
		-2	1	1/4
MSK	2	2	1	1/2
		-1	1	1/4

Evaluation of the SEP can be accomplished using the techniques of [12] by substituting the expression for γ_{MRC} directly in terms of the physical branch variables and averaging their joint p.d.f. $f_{\gamma_N}(\{\gamma_i\}_{i=1}^N)$, which gives

$$P_e = \sum_{k=1}^K \int_0^{\Theta_k} a_k(\theta) \int_0^\infty \int_0^\infty \cdots \int_0^\infty \exp\left(-\phi_k(\theta) \sum_{i=1}^N \gamma_i\right) \times f_{\gamma_N}\left(\{\gamma_i\}_{i=1}^N\right) d\gamma_N \cdots d\gamma_2 d\gamma_1 d\theta. \tag{4}$$

Note in (4) that, since the physical branches are *correlated*, direct use of the method given in [12] requires an N-fold integration. This can be alleviated by expressing γ_{MRC} in terms of the virtual branch variables as:

$$\begin{aligned} P_e &= \sum_{k=1}^K \int_0^{\Theta_k} a_k(\theta) \mathbb{E}_{\{V_n\}} \left\{ \exp\left(-\phi_k(\theta) \sum_{l=1}^L \lambda_l V_l\right)\right\} d\theta \\ &= \sum_{k=1}^K \int_0^{\Theta_k} a_k(\theta) \prod_{l=1}^L \psi_{V_l}(-\phi_k(\theta)\lambda_l) \, d\theta, \end{aligned} \tag{5}$$

where we have used the fact that the V_l's are independent. The effectiveness of the virtual branch technique is apparent by observing that the expectation operation in the above equation no longer requires an N-fold integration.

Substituting the c.f. of V_l into (5) gives

$$P_e = \sum_{k=1}^K \int_0^{\Theta_k} a_k(\theta) \prod_{l=1}^L \left[\frac{1}{1 + 2\phi_k(\theta)\lambda_l}\right]^{\mu_l/2} d\theta. \tag{6}$$

Thus the derivation of the SEP using N-branch MRC in correlated Nakagami fading reduces to a single integral with finite limits, where the integrand is an L-fold product of simple expressions.

IV. Special Cases

In this section, we study some special cases of the results obtained in the previous section. For each case, we list the eigenvalues and corresponding multiplicities, which can be substituted into (6) to obtain the SEP.

A. Single-Branch Reception (Without Diversity)

For single-branch reception ($N = 1$), K_X is a $2m_1 \times 2m_1$ diagonal matrix having only one eigenvalue ($L = 1$) given by $\lambda_1 = \mathbb{E}\{x_{1,j}^2\} = \Gamma_1 / 2m_1$ with multiplicity $\mu_1 = 2m_1$.

B. Independent Channels

All Equal Γ_i/m_i*:* In this case, K_X is a $D_T \times D_T$ diagonal matrix having only one eigenvalue ($L = 1$) given by $\lambda_1 = \Gamma_1/2m_1$ with multiplicity $\mu_1 = D_T$. Note that this includes the condition $m_i = m$ and $\Gamma_i = \Gamma$.

All Distinct Γ_i/m_i*:* In this case, K_X has N distinct eigenvalues given by $\lambda_l = \Gamma_l/2m_l$, $l = 1, \cdots, N$, each with multiplicity $\mu_l = 2m_l$. Note that this includes equal m_i's but distinct Γ_i's.[4]

C. Correlated Channels

Dual-Branch Diversity: Consider dual-branch MRC with parameter pairs (m_1, Γ_1) and (m_2, Γ_2) and covariance $\rho_{\gamma_1\gamma_2} = \sqrt{(m_1/m_2)}\rho^2$. In general, there are three distinct eigenvalues ($L = 3$) given by

$$\lambda_1, \lambda_2 = \frac{1}{2}\left(\frac{\Gamma_1}{2m_1} + \frac{\Gamma_2}{2m_2}\right) \pm \frac{1}{2}\sqrt{\left(\frac{\Gamma_1}{2m_1} + \frac{\Gamma_2}{2m_2}\right)^2 - \frac{\Gamma_1}{m_1}\frac{\Gamma_2}{m_2}(1-\rho^2)}$$

with multiplicities $\mu_1 = \mu_2 = 2m_1$, and $\lambda_3 = \Gamma_2/2m_2$ with multiplicity $\mu_3 = 2(m_2 - m_1)$.

Identical Channels with Arbitrary Correlation: Consider N branch MRC with arbitrary $\rho_{\gamma_i\gamma_j}$, where the m_i's and Γ_i's are equal for all branches. This includes the two special cases considered in [4] with two specific correlation models, namely equal correlation and exponential correlation. In this case, the $\{\lambda_l\}$'s are the L *distinct* eigenvalues of the block-diagonal covariance matrix having $2m$ identical $N \times N$ diagonal blocks given by

$$K = \frac{\Gamma}{2m}\begin{bmatrix} 1 & \rho_{1,2} & \cdots & \rho_{1,N} \\ \rho_{1,2} & 1 & \cdots & \rho_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1,N} & \rho_{2,N} & \cdots & 1 \end{bmatrix}. \quad (7)$$

where $\rho_{i,j}$ are determined from a given $\rho_{\gamma_i\gamma_j}$ by (2).

V. Numerical Examples and Conclusions

The SEP results for coherent detection of QPSK using N-branch MRC in Nakagami fading are graphically displayed in Fig. 1. Fig. 1(a) displays the case for i.i.d. Nakagami channels with equal m, as well as Γ, among all branches. Fig. 1(b) depicts the case of identical Nakagami channels with exponential correlation, where the correlation matrix given by (7) has elements $\rho_{i,j} = \rho^{|i-j|/2}$.

In conclusion, we derived the exact SEP using the virtual branch technique for coherent detection of several types of M-ary modulation with MRC in arbitrarily correlated Nakagami fading channels. This work obviates the need for lengthy derivations in separate scenarios with different numbers of diversity branches and correlation models, as our results give a simple prescription for evaluating the exact SEP performance. The results extend previously-derived results to cover numerous additional useful cases.

[4]This case was analyzed in [3] by approximating the sum of the squares of Nakagami r.v.'s by a single Nakagami r.v. with appropriate parameters. This approximation becomes exact when the Γ_i's are all equal.

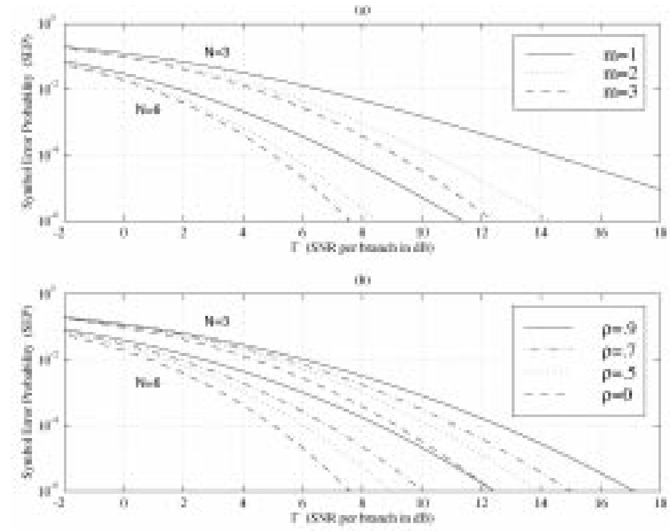

Fig. 1. The symbol error probability for QPSK with MRC as a function of the average SNR per branch for: (a) i.i.d. Nakagami channels with $m = 1, 2$ and 3; (b) identical Nakagami channels with exponential correlation where the correlation coefficient $\rho = 0, 0.5, 0.7$ and 0.9 with $m = 3$.

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