Analysis of Hybrid Selection/Maximal-Ratio Combining in Rayleigh Fading

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Abstract— In this paper, we use a novel virtual branch technique to succinctly derive the mean and variance of the combiner output signal-to-noise ratio for hybrid selection/maximal-ratio combining in a multipath-fading environment.

Index Terms— Diversity methods, fading channels, maximal-ratio combining, selection diversity, virtural branch technique.

I. INTRODUCITON

HYBRID selection/maximal-ratio combining (H-S/MRC) is a reduced-complexity diversity combining scheme, where L (with the largest signal-to-noise ratio (SNR) at each instant) out of N diversity branches are selected and combined using MRC [1]–[6]. This technique provides improved performance over L branch MRC when additional diversity is available, without requiring additional electronics and/or power consumption.

In this paper, we derive the mean and variance of the combiner output SNR of H-S/MRC for *any* L and N under the assumption of independent Rayleigh fading on each diversity branch with equal SNR averaged over the fading.[1] This is performed using a novel "virtual branch" technique, which simplifies the derivation of the mean (derived previously [1]) and permits derivation of the variance.[2]

II. DIVERSITY COMBINING ANALYSIS

A. Virtual Branch Technique: The Key Idea

The analysis of H-S/MRC based on a chosen ordering of the branches at first appears to be complicated, since the SNR statistics of the ordered branches are *not* independent. Even the *average* combiner output SNR calculation alone can require a lengthy derivation, as seen in [1]. Here, we alleviate this problem by transforming the ordered-branch variables into a new set of independently, identically distributed (i.i.d.) *virtual branches*. This reduces the derivation of the moments of the combiner output SNR to the calculation of the moments of the linear combination of i.i.d. random variables.

[1] We assume that instantaneous channel estimation using a scanning receiver across all possible diversity branches is feasible, such as with slow fading. However, H-S/MRC also offers improvement in fast fading conditions, and our results serve as an upper bound on the performance when perfect channel estimates are not available.

[2] We extend this study to analyze the symbol error probability performance in [2]–[6].

In this framework, the *average* SNR of the combined output is obtained in a more concise manner than the derivation given in [1]. Furthermore, the extension to the derivation of the combiner output SNR variance can be made using this virtual branch technique. Note that the mean and variance of the combiner output SNR for the selection diversity (SD) and MRC are special cases of our results.

B. General Theory

Let γ_i denote the instantaneous SNR of the ith diversity branch defined by

$$\gamma_i \triangleq \alpha_i^2 \frac{E_s}{N_{0i}} \tag{1}$$

where E_s is the average symbol energy, and α_i is the instantaneous gain, and N_{0i} is the noise power spectral density of the ith branch. We model the γ_i's as continuous random variables with probability density function (pdf) $f_{\gamma_i}(x)$ and mean $\Gamma_i = \mathbb{E}\{\gamma_i\}$.

Let us first consider a general diversity-combining (GDC) system with the instantaneous output SNR of the form

$$\gamma_{\text{GDC}} = \sum_{i=1}^{N} a_i \gamma_{(i)} \tag{2}$$

where $a_i \in \{0, 1\}$, $\gamma_{(i)}$ is the ordered γ_i, i.e., $\gamma_{(1)} > \gamma_{(2)} > \cdots > \gamma_{(N)}$, and N is the number of available diversity branches. It will be apparent later that several diversity combining schemes, including H-S/MRC, turn out to be special cases of (2).

We assume that the instantaneous branch SNR's γ_i are independent with the same average SNR, i.e., $\Gamma_i = \Gamma$ for $i = 1, \cdots, N$. Denoting $\boldsymbol{\gamma}_{(N)} \triangleq [\gamma_{(1)}, \gamma_{(2)}, \cdots, \gamma_{(N)}]^T$, where the superscript T denotes transpose, the joint pdf of $\gamma_{(1)}, \gamma_{(2)}, \cdots, \gamma_{(N)}$ for a Rayleigh fading channel can be derived using the theory of "order statistics" [7] as

$$f_{\boldsymbol{\gamma}_{(N)}}(\{\gamma_{(i)}\}_{i=1}^N)$$
$$= \begin{cases} \dfrac{N!}{\Gamma^N} e^{-(1/\Gamma) \Sigma_{m=1}^N \gamma_{(m)}}, & \gamma_{(1)} > \gamma_{(2)} > \cdots > \gamma_{(N)} > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

The mean SNR of combiner output signal is given by (4), shown at the bottom of the next page.

Since the statistics of the ordered branches are *no* longer independent, the evaluation of the mean SNR involves nested integrals, which are in general cumbersome and complicated to compute. This can be alleviated by transforming the instantaneous SNR's of the ordered diversity branches into a

new set of *virtual branch* instantaneous SNR's, V_n, using the following relation:

$$\gamma_{(i)} = \sum_{n=i}^{N} \frac{\Gamma}{n} V_n. \tag{5}$$

The Jacobian J of the above transformation is $\Gamma^N/N!$. Let $\gamma_{(N+1)} = 0$. Then the recursion for $\gamma_{(i)}$ is

$$\gamma_{(i)} = \gamma_{(i+1)} + \frac{\Gamma}{i} V_i, \qquad i = 1, \cdots, N. \tag{6}$$

Hence, the V_n's are not necessarily ordered, and $0 < V_n < \infty$.

Denoting $\boldsymbol{V}_N \triangleq [V_1, V_2, \cdots, V_N]^T$, the joint pdf of $V_1, V_2, \cdots, V_N$ can be obtained, using the distribution theory for transformations of random vectors [7], as

$$f_{\boldsymbol{V}_N}(\{v_n\}_{n=1}^{N}) = J f_{\boldsymbol{\gamma}_{(N)}}(\{\gamma_{(i)}\}_{i=1}^{N})|_{\gamma_{(i)} = \Sigma_{n=1}^{N}(\Gamma/n)v_n}. \tag{7}$$

By substituting (5) into (3) along with J, it can be verified after some algebra that (7) becomes

$$f_{\boldsymbol{V}_N}(\{v_n\}_{n=1}^{N}) = \prod_{n=1}^{N} f_{V_n}(v_n) \tag{8}$$

where $f_{V_n}(\cdot)$ is the pdf of V_n given by

$$f_{V_n}(v) = \begin{cases} e^{-v}, & 0 < v < \infty \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

Therefore, the instantaneous SNR's of the virtual branches are i.i.d. normalized exponential random variables.[3]

The instantaneous SNR of the combiner output can now be expressed in terms of the instantaneous SNR of the virtual branches as

$$\gamma_{\text{GDC}} = \sum_{n=1}^{N} b_n V_n \tag{10}$$

where $b_n = (\Gamma/n) \sum_{i=1}^{n} a_i$. Using the fact that normalized exponential random variables have unity mean ($\mathbb{E}\{V_n\} = 1$), the mean of the combiner output SNR can now be calculated as

$$\Gamma_{\text{GDC}} = \mathbb{E}\left\{\sum_{n=1}^{N} b_n V_n\right\} = \sum_{n=1}^{N} b_n. \tag{11}$$

Similarly, the variance of the combiner output SNR is

$$\sigma_{\text{GDC}}^2 = \text{Var}\left\{\sum_{n=1}^{N} b_n V_n\right\} = \sum_{n=1}^{N} b_n^2 \tag{12}$$

where the unit variance of the normalized exponential random variable ($\text{var}\{V_n\} = 1$) is used. Note that the independence of the virtual branch variables plays a key role in simplifying the derivation of (11) and (12).

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[3] When the Γ_i's are not all equal, then the virtual branches are only *conditionally* independent. This is investigated in [2].

The average SNR gain of the diversity combining compared to a single branch system is a commonly accepted performance measure [8]. This quantity is calculated as

$$G_{\text{GDC}} \triangleq 10 \log_{10}\left\{\frac{\Gamma_{\text{GDC}}}{\Gamma}\right\} = 10 \log_{10}\left\{\frac{\sum_{n=1}^{N} b_n}{\Gamma}\right\}. \tag{13}$$

To assess the effectiveness of diversity combining in the presence of multipath, we also define the normalized standard deviation of the combiner output SNR as

$$\sigma_{\text{n,GDC}} \triangleq 10 \log_{10}\left\{\frac{\sqrt{\sigma_{\text{GDC}}^2}}{\Gamma_{\text{GDC}}}\right\} = 10 \log_{10}\left\{\frac{\sqrt{\sum_{n=1}^{N} b_n^2}}{\sum_{n=1}^{N} b_n}\right\}. \tag{14}$$

C. H-S/MRC Analysis

In this section, the general theory derived in Section II-B is used to evaluate the performance of H-S/MRC. The instantaneous output SNR of H-S/MRC is

$$\gamma_{\text{HS/MRC}} = \sum_{i=1}^{L} \gamma_{(i)} \tag{15}$$

where $1 \leq L \leq N$. Note that $\gamma_{\text{HS/MRC}} = \gamma_{\text{GDC}}$ with $a_i = 1$, for $i = 1, \cdots, L$ and $a_i = 0$ for $i = L+1, \cdots, N$. In this case, $b_n = \Gamma$ for $n = 1, \cdots, L$ and $b_n = \Gamma(L/n)$ for $n = L+1, \cdots, N$.

Substituting this into (11) and (12), the mean and the variance of the combiner output SNR can be easily obtained as

$$\Gamma_{\text{HS/MRC}} = L\left(1 + \sum_{n=L+1}^{N} \frac{1}{n}\right)\Gamma \tag{16}$$

and

$$\sigma_{\text{HS/MRC}}^2 = L\left(1 + L \sum_{n=L+1}^{N} \frac{1}{n^2}\right)\Gamma^2 \tag{17}$$

respectively. Therefore, the average SNR gain of H-S/MRC in decibels is

$$G_{\text{HS/MRC}} = 10 \log_{10}\left\{L\left(1 + \sum_{n=L+1}^{N} \frac{1}{n}\right)\right\} \tag{18}$$

$$\Gamma_{\text{GDC}} = \mathbb{E}\{\gamma_{\text{GDC}}\} = \int_0^{\infty} \int_0^{\gamma_{(1)}} \cdots \int_0^{\gamma_{(N-1)}} \sum_{i=1}^{N} a_i \gamma_{(i)} f_{\boldsymbol{\gamma}_{(N)}}(\{\gamma_{(i)}\}_{i=1}^{N}) \, d\gamma_{(N)} \cdots d\gamma_{(2)} \, d\gamma_{(1)}. \tag{4}$$
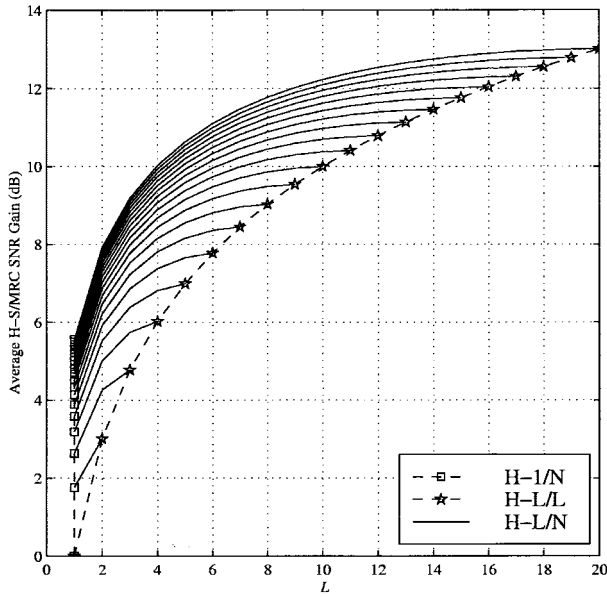
Fig. 1. The average SNR gain with H-L/N as a function of L for various N. The solid curves are parameterized by different N, starting from the lowest curve with $N = 2$, and increase monotonically to the highest curve with $N = 20$.
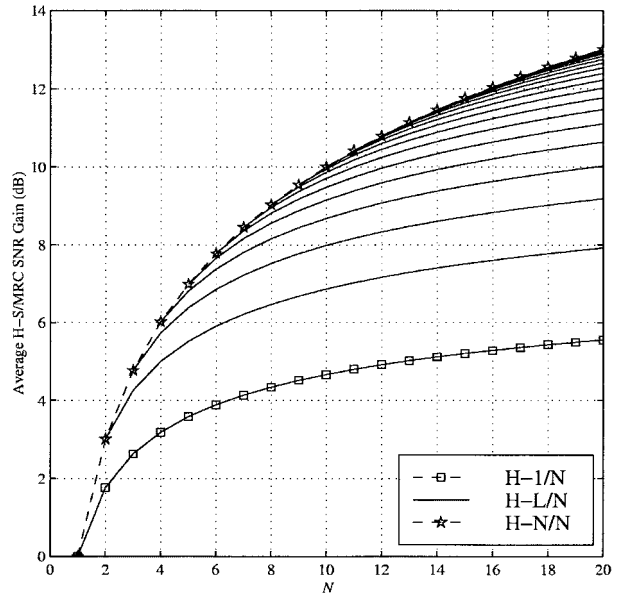

Fig. 2. The average SNR gain with H-L/N as a function of N for various L. The lowest solid curve is for $L = 1$, and L increases monotonically to the highest solid curve with $L = 20$.

and normalized standard deviation of the combiner output SNR in decibels is

$$
\sigma_{\mathrm{n,HS/MRC}} = 10 \log_{10} \left\{ \frac{\sqrt{\left(1 + L \sum_{n=L+1}^{N} \frac{1}{n^2}\right)}}{\sqrt{L}\left(1 + \sum_{n=L+1}^{N} \frac{1}{n}\right)} \right\}. \quad (19)
$$

Note that (16) agrees with the result of [1]. Also note that SD and MRC are limiting cases of H-S/MRC with $L = 1$ and $L = N$, respectively, and the result of (16) agrees with the well-known results for SD and MRC given by [8, pp. 313–319, eqs. (5.2.8), (5.2.16)].

III. NUMERICAL EXAMPLES

In this section, the results derived in the previous section for H-S/MRC are illustrated. The notation H-L/N is used to denote H-S/MRC that selects and combines L out of N branches. Note that H-1/1 is a single branch receiver, H-1/N is SD, and H-N/N is MRC with N branches.

Fig. 1 shows the average SNR gain with H-L/N as a function of L for various N. The data points denoted by the "squares" represent the gain using H-1/N. The data points denoted by the "stars" represent the gain using H-L/L and serve as a lower bound for the gain of H-L/N. It can be seen that H-L/N can provide a significant gain over H-L/L, when additional diversity is available. Fig. 2 shows the average SNR gain using H-L/N versus N for various L. The data points denoted by the "squares" and "stars" represent the average SNR gain using H-1/N and H-N/N, respectively. Note that the curves for H-1/N and H-N/N serve, respectively, as a lower and upper bound for the average SNR gain of H-L/N.
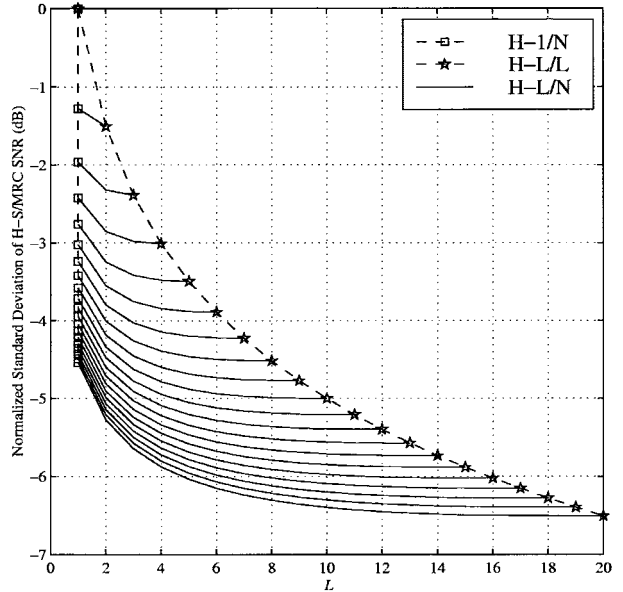

Fig. 3. The normalized standard deviation of the H-L/N combiner output SNR as a function of L for various N. The solid curves are parameterized by different N, starting from the highest curve with $N = 2$, and decrease monotonically to the lowest curve with $N = 20$.

Fig. 3 shows the normalized standard deviation of the H-L/N combiner output SNR as a function of L for various N. The normalized standard deviation of the combiner output SNR for H-1/N and H-L/L can be seen as the two limiting cases. The curve for H-L/L upper bounds the normalized standard deviation of the H-L/N combiner output SNR. Note that H-L/N can provide significant reduction in the normalized standard deviation of the combiner output SNR as compared to H-L/L, when additional diversity is available. Fig. 4 shows the normalized standard deviation of the H-L/N combiner output SNR versus N for various L. It can be clearly
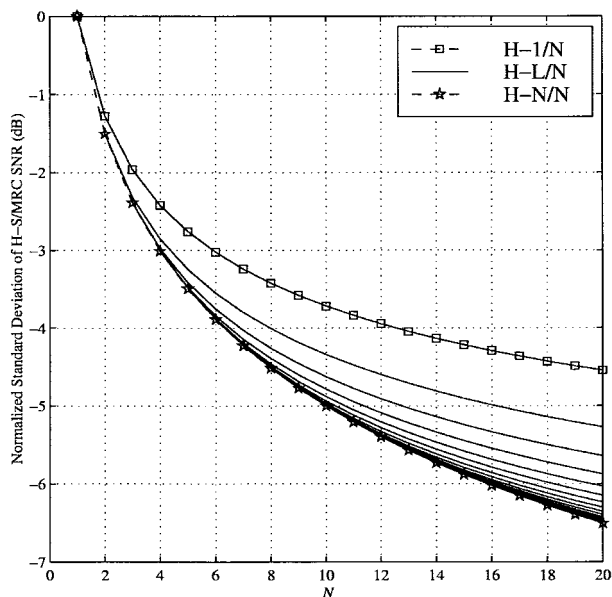
Fig. 4. The normalized standard deviation of the H-L/N combiner output SNR as a function of N for various L. The highest solid curve is for $L = 1$, and L decreases monotonically to the lowest solid curve with $L = 20$.

seen that the curves for H-1/N and H-N/N upper and lower bound, respectively, the normalized standard deviation of the H-L/N combiner output SNR.

IV. CONCLUSIONS

We derived the mean and variance of the combiner output SNR of H-S/MRC in a multipath fading environment for *any*

L and N. We analyzed this system using a novel "virtual branch" technique, which resulted in a simple derivation of the mean (derived previously [1]) and permitted derivation of the variance.

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REFERENCES

[1] N. Kong and L. B. Milstein, "Combined average SNR of a generalized diversity selection combining scheme," in *Proc. IEEE Int. Conf. Commun.*, Atlanta, GA, June 1998, vol. 3, pp. 1556–1560.
[2] M. Z. Win and J. H. Winters, "Analysis of hybrid selection/maximal-ratio combining of diversity branches with unequal SNR in Rayleigh fading," in *Proc. 49th Annu. Int. Veh. Technol. Conf.*, Houston, TX, May 1999, vol. 1, pp. 215–220.
[3] M. Z. Win, R. K. Mallik, G. Chrisikos, and J. H. Winters, "Canonical expressions for the error probability performance for M-ary modulation with hybrid selection/maximal-ratio combining in Rayleigh fading," in *Proc. IEEE Wireless Commun. Networking Conf.*, New Orleans, LA, Sept. 1999, vol. 1, pp. 266–270.
[4] M. Z. Win, G. Chrisikos, and J. H. Winters, "Error probability performance for M-ary modulation in Rayleigh fading using hybrid selection/maximal-ratio combining," in *Proc. Military Commun. Conf.*, Atlantic City, NJ, Nov. 1999.
[5] M. Z. Win and J. H. Winters, "Exact error probability expressions for hybrid selection/maximal-ratio combining in Rayleigh fading," in *Proc. IEEE Global Telecommun. Conf.*, Rio de Janeiro, Brazil, Dec. 1999.
[6] ———, "Virtual branch analysis of symbol error probability for hybrid selection/maximal-ratio combining in Rayleigh fading," *IEEE Trans. Commun.*, submitted for publication.
[7] P. J. Bickel and K. Doksum, *Mathematical Statistics: Basic Ideas and Selected Topics*, 1st ed.   Oakland, CA: Holden-Day, 1977.
[8] W. C. Jakes, Ed., *Microwave Mobile Communications*.   Piscataway, NJ: IEEE Press, 1995, reissue.